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Optimal Auctions: The Efficiency of Oral Auctions
without Reserve for Risk Neutral Bidders with Private
Values and Costly Information

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Optimal Auctions: The Efficiency of Oral Auctions Without
Reserve for Risk Neutral Bidders with Private Values
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ABSTRACT

We argue that for one family of auction models, oral auctions generate nearly as high a selling price as can be reasonably expected. Specifically, consider the case of uninformed risk neutral individuals who must acquire (at a known cost) perfect information on their own values for the object being sold before being allowed to bid; perhaps one incurs costs in travelling to the auction, but once there can costlessly inspect and accurately appraise the object(s) being sold. To assure the potential buyers as a whole a non-negative expected profit, the selling price cannot average above the maximum (over sets of potential buyers) of the bidders' highest value net of all potential buyer's costs. Then for a progressive oral auction with a reservation price equal to the seller's value for the object, at equilibrium--with respect both to who participates and to how bidders bid--the expected price of the object falls short of the maximum possible by less than the effect on the selling price of having one too many potential buyers.

At entry fee (paid to the seller), a reservation price, or a subsidy (paid by the seller) can decrease this gap between the equilibrium price and the maximum possible. However, the optimal sizes of such fees, reservation prices, or subsidies depend on the distribution of the potential buyers' values; the auction must now be tailored to the specific situation. Still, an optimal fee or reservation price will not reduce the number of potential buyers below the original equilibrium number, nor will an optimal subsidy increase the number by more than one. Thus, roughly speaking, except for the discreteness of the number of potential buyers, an oral auction without reserve generates the highest selling price that can be reasonably expected in the case of risk neutral bidders, private values, and costly participation.

Introduction:

In certain auction model with a fixed number of bidders, the seller would maximize the expected selling price of the object being sold by setting a reservation price in excess of his own value for the object, a reservation price that results in a positive probability of not selling the object even though at least one bidder made a favorable offer for it. For example, consider the case of n bidders, each of which knows his own value for the object precisely and knows that the bidders' values were drawn independently from some cumulative probability distribution $F(\cdot)$. Imagine that the object will be sold through a sealed bid auction with a reservation price of r ; in effect, the seller bids r . The highest bidder wins the object, but only pays an amount equal to the second highest bid; this second price mechanism approximates oral auctions. No bidder can do better than bid equal to his own value, and if all bidders do bid equal to their respective values, this auction generates an expected selling price of $r n (1-F(r)) F^{n-1}(r) + \int_{x=r}^{\infty} x n (n-1) (1-F(x)) F^{n-2}(x) dF(x)$. Differentiating this expression with respect to the reservation price r , and setting the resulting expression equal to zero gives the first order condition of $1-F(r) = r dF(r)$ for an optimal reservation price. Specifically, for the case of the standard uniform distribution, the optimal reservation price equals one half, and the seller retains the object with probability $(1/2)^n$. (The same first order condition arises even if the number of bidders were a random variable with some fixed distribution.)

The work of Myerson (1981) implies that no other mechanism that guarantees each bidder a non-negative expected profit can generate a

higher equilibrium selling price for the case of a fixed number of risk neutral bidders with independently distributed, privately known values (and certain regularity on the distribution of the values) than the second price sealed bid auction with an optimally chosen reservation price. A first price sealed bid auction with an appropriate reservation price, and many other mechanisms, generate an equally high expected selling price. Still, all these expected selling price maximizing mechanisms result in a positive probability of the seller keeping the object even though at least one bidder made a favorable offer for it.

On the other hand, practice does not always correspond to the predictions of this theory. For example, seemingly experienced auctioneers advertise "all items to be sold without reserve," or lobby for stricter penalties for being caught employing shills--agents, of the seller, that covertly affect a non-trivial reservation price. Or, for example, in the early nineteenth century, according to Albion (1961), the auctioneers of goods imported to the Port of New York, with the express intention of attracting additional buyers and of increasing the volume of goods sold, petitioned the State for legislation that would encourage selling all items offered for sale. It worked. The volume of trade at the Port of New York grew much faster than at other eastern ports, even before the Erie Canal opened.

This suggests, at least to this researcher, that the existing theory must be flawed. Either the existing models fail to capture, or inaccurately represent, something of importance in practical markets,

or we solve the models incorrectly and obtain incorrect conclusions. But what is the problem? The rules of actual auctions appear simple enough to describe. The number and types of objects being offered appear equally simple to describe. The possibility of bidders bidding to maximize their respective expected profits or utilities also appears plausible. But what about the actual number of bidders and how much information they have?

The Port of New York example suggests that the number of bidders attending a particular type of auction might depend on the auction type. More bidders might be attracted to an auction of a type in which the bidders as a whole typically fare quite well, than to a less promising auction. As argued by the New York auctioneers, the lower the reservation price, the more attractive the auction, and the more buyers will attend--the auctioneers hoped to make up through increased volume for any reduction of profit in individual sales.

The existing theory, for instance as surveyed by Engelbrecht-Wiggans (1980) and McAfee and McMillan (1985), or as contributed by Vickrey (1961), Myerson (1981), and Milgrom and Weber (1982), derives mainly from models with an exogenously fixed number of bidders each of which obtains an exogenously fixed amount and type of information. A few models--most notably, those of Matthews (1984) and Lee (1984)--do endogenise the acquisition of information. However, additional bidders may mean more than just additional information being present at the auction. In particular, one of the additional bidders may value the object more highly than any of the original bidders, and thus the

market value of the object--as presumably happened with goods imported to New York--increases with the number of buyers attending the auction.

In fact, Engelbrecht-Wiggans (1986) has already illustrated that in the case of risk neutral bidders with uniformly distributed private values, raising the reservation price in a sealed bid second price auction to the point of loosing a bidder costs the seller more than can be gained from a non-trivial reservation price. For the case of a Poisson distributed number of bidders with the mean number of bidders set so that increasing it further would drive bidders' profits negative, the optimal reservation price equals the seller's value for the object being sold. The current paper examines the efficiency of oral auctions more generally under the presumption that the profitability of the auction to the bidders might effect the number of bidders.

We start with a couple of concrete, though fictitious, settings in which the number of potential buyers might reasonably vary with the profitability of the auction; these examples provide the basis for subsequent definitions and models. In the "art market" example, imagine a collection of risk neutral art dealers who have certain annual fixed costs of being in business. Every so often, someone offers to sell a rare painting at auction. Conditional on any common public knowledge, each dealer has some private information on his own resale value for the painting. Each art dealer must cover his fixed costs of being in business from his profits on reselling paintings bought at auction. Then, ignoring the costs of becoming an art dealer, or of quitting the business, the number of art dealers might adjust itself until each

dealer can still cover this fixed costs, but no additional individual could become a dealer and hope to cover his fixed costs; the number of dealers, and therefore the number of potential buyers for art sold at auction, depends on the profitability of art auctions to the potential buyers.

Alternatively, in the "estate sale" example, imagine that a collection of potential buyers regularly scan the local newspaper's notices of upcoming auctions. A specific individual attends a particular sale only if the estate to be sold, as described by the notice, includes items of interest to this particular individual; this example differs from the previous one in that only a subset of the potential buyers actually bid in any particular auction, a subset that will be modelled as being randomly selected. Assume that until an individual has expended the time and energy needed to attend the auction, he has only the same information as is available to all other potential buyers as to the specific characteristics--and, therefore, as to the value to this individual--of the items to be sold. However, once at the auction, an individual may costlessly inspect and accurately appraise each object. As with the number of art dealers in the previous example, here the number of potential buyers who scan the auction notices may depend on the profitability of attending such auctions; those who scan the notices presumably expect a non-negative net gain from doing so, while those who don't scan the notices presumably expect a positive loss if they were to get into the habit.

The next section defines a model of oral auctions with an endogenously determined number of risk neutral bidders and private, though not

necessarily independent, information. We then derive and examine an equilibrium to this model. The equilibrium set of bidders happens to be the set that maximizes the net value of the object, in other words, the equilibrium optimizes the tradeoff between the costs and benefits of an additional potential buyer. At equilibrium--with respect to who becomes a potential buyer and to how actual bidders bid--the expected price in the oral auction without reserve falls short of the maximum that any mechanism that guarantees bidders a non-negative expected profit can by less than the effect on the selling price of one potential buyer beyond the equilibrium number.

The expected selling price might be improved slightly through an appropriate reservation price, entry fee, or subsidy. The optimal size of such improvements depend on the situation. But, the improvement will not decrease the equilibrium number of bidders, nor increase it by more than one, from the original number. Thus, roughly speaking, except for the discreteness of the number of potential buyers, the oral auction without reserve--an auction not tailored to specific cases--generates as high an expected selling price as can be reasonably expected in cases of risk neutral bidders with private values and costly participation.

The Model Defined

As suggested by the estate sale example, define the following three different sets of individuals: 1) the set N_0 of potential risk neutral participants--those who subscribe to the paper carrying auction notices, 2) the set N_1 of potential buyers--the subset of potential participants who actually read the auction notices, and 3) the set N_2 of

actual bidders--the subset of the potential buyers who attend a particular sale (in the art market example, the set of potential buyers would be the same as the set of actual bidders). A potential buyer i has no say himself as to whether or not he becomes an actual bidder; i becomes an actual bidder with probability p_i , upon which he pays c_i , in exchange for which he sees the outcome x_i of a random variable X_i . We write \underline{x} and \underline{X} for the vectors (x_1, x_2, \dots) and (X_1, X_2, \dots) . For simplicity, we think of there being an x_i for each potential participant i even though only the actual bidders have any use for, or any need to see--their respective x_i 's. Then, let $v_i(\underline{x})$ denote the expected value, to i , of the object conditional on $\underline{X} = \underline{x}$. Although the seller may have a non-zero value for the object, we can measure the potential buyers' values with respect to the seller's value, and therefore assume, without loss of generality, that the seller has a value of zero for the object.

Assume that the actual bidders are selected from the set of potential buyers independently of each other and independently of \underline{x} and the true state z of Nature. More precisely, assume the $\Pr(N_2 | N_1)$ --the probability of the set of actual bidders being N_2 conditional on the set of the potential buyers being N_1 --equals $\prod_{j \in N_2/N_1} (1-p_j)$ for all $N_2 \subseteq N_1$, and zero otherwise. For future reference, note that $\Pr(N \cup i | N_1 \cup i) + \Pr(N \setminus i | N_1 \cup i) = p_i \Pr(N | N_1 \setminus i) + (1-p_i) \Pr(N | N_1 \setminus i) = \Pr(N | N \setminus i)$ for all $N \subseteq N_1 \cup i$.

Once the set of actual bidders has been set, Nature selects the true state z according to the cumulative probability distribution $H(\cdot)$, selects each x_i , independently of everything except z , according to the cumulative distribution $F_i(\cdot | \cdot)$ of X_i conditional on z , and

shows the outcome x_i to (and only to) i . Note that since z may be the outcome of an arbitrary random variable--indeed the random variable may include \underline{x} as a subset of its components--the X_i 's may be assumed to be independent of each other conditional on z without any loss of generality in the joint distribution of the X_i 's.

Now the object will be auctioned by starting the asking price at the reservation price r . At each asking price, each bidder indicates "yes" or "no" to the question "would you be willing to pay the current asking price?" (A bidder need not be truthful, and may even say "yes" after having previously said "no" at a lower price.) If no one says "yes" at the starting price, then the seller keeps the object. Otherwise, the asking price rises continuously until the first instant that fewer than two bidders say "yes." At that point the last bidder to say "yes"--or randomly selected last bidder in the case of ties--wins the object and pays the current asking price. The auction is over!

In fact, we have yet to see an auction conducted according to these rules. However, the outcome of an auction conducted according to these rules approximates the outcome of an actual oral auction. Therefore, these rules provide an interesting, if implausible, model of actual auction rules.

Note also that the rules do not specify what any one bidder might learn about others' answers at each asking price. We eventually assume that $v_i(\underline{x})$ depends on \underline{x} only through x_i ; nothing that i could learn about others' x_j 's would have any effect of i 's estimated expected value for the object. While this may be a slightly non-standard definition of "private(ly known) values," it captures the

critical aspects, and we adopt it as our definition. In the case of private values, it does not matter what one bidder might learn about others' information through their bidding, so we simply omit specifying what any one bidder sees about others' bidding.

Finally, to complete the model's definition, we describe how potential participants become potential buyers. Imagine that the potential participants line up in order of increasing cost c_j . Then, one by one, each individual decides whether or not to become a potential buyer. An individual, when making this decision, knows who has already joined the set of potential buyers, and knows the entire model description; only the actual \underline{x} and z , and the final sets N_1 and N_2 remain uncertain. Presumably, an individual decides to become a potential buyer if and only if he expects a non-negative profit, net of all costs, from being a member of the final, as yet uncertain, set of potential buyers.

The "market value" of an object provides a useful yardstick for measuring the efficiency of an auction. Roughly speaking, the market value $V(N_1)$ equals the expected value of the object to the bidder who values it most highly. More precisely, $V(N_1) = \sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) E[\max_{j \in N_2} v_j(\underline{x})]$, which we occasionally write as

$$\sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) \int \int \max_{j \in N_2} v_j(\underline{x}) \prod_{j \in N_0} dF_j(x_j | z) dH(z).$$

Since an individual presumably becomes a potential buyer if and only if his expected net profit as a potential buyer is non-negative, the seller cannot reasonably expect a selling price in excess of the market value reduced by the participation costs of the potential buyers.

Specifically, define the "net (social) value" of the object to be

$$V(N_1) = \sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) \sum_{j \in N_2} c_j. \text{ Of course, since the net value}$$

value depends on the set of potential buyers, some sets N_1 may result in higher net values than others. Although the seller has no direct control over the set of potential buyers, the individuals' choices on becoming a potential buyer may be effected by the reservation price, or other auction details, selected by the seller. It turns out that in our model, the equilibrium set N_1 maximizes the net value over all possible subsets N_1 of the potential participants.

The optimality of the equilibrium set of potential buyers will arise from the fact that in our model, the marginal contribution of an additional potential seller to the object's market value equals that individual's expected profit from being a potential buyer; in his own self interest, an individual becomes a potential buyer if and only if so doing increases the object's net value.

To state the pivotal fact more precisely, start by defining

$$\Delta(\underline{x}, i, N_2) = \max_{j \in N_2} v_j(\underline{x}) - \max_{j \in N_2 \setminus i} v_j(\underline{x}), \text{ which reduces to zero if } i \in N_2 \text{ or}$$

$$\text{if } v_i(\underline{x}) < \max_{j \in N_2 \setminus i} v_j(\underline{x}), \text{ and to } v_i(\underline{x}) - \max_{j \in N_2 \setminus i} v_j(\underline{x}) \text{ otherwise; for an}$$

appropriately defined second price auction, this would be the expected equilibrium profit to i , conditioned on $\underline{X} = \underline{x}$, when bidding against the set of bidders of N_2 . However, we don't allow potential buyers to themselves decide whether or not to actually bid--an individual decides whether or not to join the set N_1 , and does so without any private information, but then Nature chooses N_2 from N_1 . Thus, we look at the expected value of $\Delta(\underline{X}, i, N_2)$ conditional on N_1 , and obtain the following theorem:

Theorem 1: $E[\Delta(\underline{x}, i, N_2) | N_1] = V(N_1) - V(N_1 \setminus i)$ for all i in N_0 .

(Note that the right hand side is zero unless $i \in N_1$.)

Proof: For $i \in N_1$, $E[\Delta(\underline{x}, i, N_2) | N_1]$

$$\begin{aligned}
 &= \sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) \int \int_{\underline{z} \leq \underline{x}} \Delta(\underline{x}, i, N_2) \prod_{j \in N_0} dF_j(x_j | z) dH(z) \\
 &= \sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) \int \int_{\underline{z} \leq \underline{x}} \left(\max_{j \in N_2} v_j(\underline{x}) - \max_{j \in N_2 \setminus i} v_j(\underline{x}) \right) \prod_{j \in N_0} dF_j(x_j | z) dH(z) \\
 &= \sum_{N_2 \subseteq N_1} \Pr(N_2 | N_1) \int \int_{\underline{z} \leq \underline{x}} \max_{j \in N_2} v_j(\underline{x}) \prod_{j \in N_0} dF_j(x_j | z) dH(z) \\
 &\quad - \sum_{N_2 \subseteq N_1} [\Pr(N_2 \cup i | N_1) + \Pr(N_2 | N_1)] \int \int_{\underline{z} \leq \underline{x}} \max_{j \in N_2} v_j(\underline{x}) \prod_{j \in N_0} dF_j(x_j | z) dH(z)
 \end{aligned}$$

(using the relationship noted when we defined the p_j 's)

$$\begin{aligned}
 &= V(N_1) - \sum_{N_2 \subseteq N_1 \setminus i} \Pr(N_2 | N_1 \setminus i) \int \int_{\underline{z} \leq \underline{x}} \max_{j \in N_2} v_j(\underline{x}) \prod_{j \in N_0} dF_j(x_j | z) dH(z) \\
 &= V(N_1) - V(N_1 \setminus i) \text{ as desired}
 \end{aligned}$$

On the other hand, for $i \notin N_1$, $i \notin N_2$ for all $N_2 \subseteq N_1$. Thus, $\Delta(\underline{x}, i, N_2) = 0$ for all \underline{x} and all $N_2 \subseteq N_1$, implying that $E[\Delta(\underline{x}, i, N_2) | N_1] = 0$ as needed to complete the proof.

Note that so far, we have not used the assumption of bidders being risk neutral (except to the extent that the assumption may be needed to make sense of the $v_j(\underline{x})$'s not being a function of the true state of Nature z). Nor did we need the bidders to have private values. In fact, Theorem 1 follows mainly from the definition of market value and

of the independence of potential buyers in becoming actual bidders. Hereafter though, we assume that the bidders have private values.

The Equilibrium Derived

Theorem 2: For any fixed set of N_1 of potential buyers, regardless of what each bidder i knows about N_0 , N_1 , $\Pr(N_2|N_1)$, $H(\cdot)$, or the $F_j(\cdot|\cdot)$'s, a bidding equilibrium results if each bidder i says "yes" if and only if v_i exceeds the current asking price.

Proof: At the hypothesized equilibrium, the price of the object to i if he wins would be $\max_{j \in N_2 \setminus i} v_j$. Thus, the winner's price does not depend on how the winner bid. However, an individual's bid does effect whether or not he wins. But, i makes a non-negative profit from winning the object at a price of $\max_{j \in N_2 \setminus i} v_j$, if and only if this price does not exceed his own value v_i for the object. Therefore, i can't do better than to continue saying "yes" until the asking price reaches his value.

Note that this is not a dominant strategy equilibrium, as it might have been in a second sealed bid auction. In particular, the benefits of bidding as prescribed depend on it not effecting others' bidding in such a way to adversely effect the expected price when i wins. Since we allow the possibility that other bidders see how i bids, the equilibrium must assure i that how he bids is not adversely reflected in others' bids; the stated equilibrium assures that.

Note also that the stated equilibrium would hold even if the asking price were raised in discrete steps. The steps need not even be equal,

so long as their size did not depend on how bidders bid. Still, this gains little realism over continuously raised prices, but has the drawback of possibly creating a gap between the selling price and the second highest valuing bidder's value for the object, thereby reducing the expected selling price and also making the subsequent results more difficult to state.

Finally, note that at this equilibrium, when i wins, i pays $\max_{j \in N_2 \setminus i} v_j$ for an object of value v_i (which happens to equal $\max_{j \in N_2} v_j$), thereby giving i a net profit of $\max_{j \in N_2} v_j - \max_{j \in N_2 \setminus i} v_j$. Then, applying Theorem 1 gives the following corollary:

Corollary: Gross of information and other participation costs, but net of anything paid for any objects won, at the truthful bidding equilibrium described above, each potential buyer i has an expected profit of $V(N_1) - V(N_1 \setminus i)$ as a result of being a member of the set N_1 of potential buyers.

Now, to characterize an equilibrium for becoming potential buyers, we need to assume a certain amount of symmetry to the model. In particular, assume all the p_j 's to be equal (to, say, p). Also assume the bidders' private values to be identically distributed (say, according to the cumulative probability distribution function $F(\cdot | \cdot)$ of the value conditional on the true state). This means that $V(N_1)$ depends only on the number of potential buyers n_1 in N_1 , and will be written as $V(n_1)$ hereafter. In addition, although we have not needed to do so far, hereafter assume all potential participants to be risk neutral if p is less than unity.

Theorem 3: An equilibrium results if each individual j decides to become a potential buyer if and only if $pc_j \leq V(n+1) - V(n)$, where n denotes the number of individuals who decided to become buyers before j ; and if he decides to become a potential buyer and happens to become an actual bidder, he bids truthfully.

Proof: Once a set of potential buyers has been fixed, Theorem 2 assures that bidding truthfully results in an equilibrium. When all actual bidders bid truthfully, each potential buyer j has an expected profit, net of all costs, of $V(n_1) - V(n_1-1) - pc_j$, where n_1 equals the final number of potential buyers. However, as n increases, $V(n) - V(n-1)$ decreases, while the marginal individual's pc_j increases. Thus, all potential buyers' expected profits decrease with the number of potential buyers. Therefore, at the hypothesized equilibrium, j ends up being a potential buyer if and only if j could expect a non-negative profit from being a potential buyer. Since no individual could do any better, we have an equilibrium.

Theorem 4: For the equilibrium set N_1 (of the n_1 individuals with the lowest c_j 's), $V(N_1) - \sum_{j \in N_1} pc_j \geq V(N) - \sum_{j \in N} pc_j$ for all $N \subseteq N_0$. In words, the equilibrium set of potential buyers maximizes the net (social) value of the object over all possible sets of potential buyers.

Proof: Follows directly from the previously observed fact that j joined N_1 if and only if doing so is profitable, together with the fact that the expected profit to the last bidder to join N_1 equals that bidder's marginal contribution to the net (social) value.

Theorem 5: If all c_j equal some c , then the maximum possible expected price from any mechanism that guarantees each bidder a non-negative expected profit will be less than the expected equilibrium price in our oral auction without reserve when one more than the equilibrium number of potential buyers participates.

Proof: With a profit of $V(n) - V(n-1)$ for each potential buyer if there n of them, and with the expected selling price equal to the market value less the total profit of the n potential buyers, the expected price must be given by $V(n) - n(V(n) - V(n-1)) = n V(n-1) - (n-1) V(n)$. Now, at the equilibrium number n^* of potential buyers, $p_c > V(n^* + 1) - V(n^*)$; otherwise the $n^* + 1$ st potential participant would have become a potential buyer. Thus, the equilibrium net (social) value $V(n^*) - n^* p_c < V(n^*) - n^*(V(n^*+1) - V(n^*))$ which equals the equilibrium price from $n = n^* + 1$ potential buyers.

This theorem establishes the oral auction without reserve as a very robust, nearly optimal auction. In particular, recall that the expected price for any mechanism that guarantees each bidder non-negative expected profit cannot exceed the net (social) value. Thus, the above theorem states that the expected equilibrium price (with the equilibrium number of potential buyers) falls short of the maximum possible expected price by less than the effect of one potential buyer beyond the equilibrium number on the equilibrium selling price in an oral auction. Since we cannot reasonably expect real world individuals to always follow a model's equilibrium precisely, we have shown that

the shortfall of the equilibrium price in our model from the maximum possible in our model is less than the amount of variation in the actual selling price that would arise from real world behavior slightly different than the model's equilibrium.

Furthermore, we might now expect the shortfall to shrink to zero as p shrinks to zero (with an offsetting increase in the number of potential buyers). In particular, for very small p , the expected effect of one too many potential buyers on the equilibrium selling price will be very small; the extra potential buyer won't actually be a bidder often enough to make much difference. (This may explain why auctioneers incur the expense of advertising what will be auctioned at each estate sale--the more detail potential buyers have, the more discriminately each can be. In effect, this drives p down.)

Note that right now, the number of actual bidders is a binomially distributed random variable with parameters p and n equal to the number of potential buyers. As p goes to zero with n increasing so that the product np stays fixed, the binomial distribution converges to a Poisson distribution. The following theorem examines directly the optimality of the equilibrium selling price in the limiting case of a Poisson distributed number of bidders.

Theorem 6: For the case of all c_j equal to some c and a Poisson distributed number of bidders with the mean number λ chosen so that any further increase would drive bidders' expected profits negative, the expected equilibrium price in our oral auction equals the maximum

possible net value, and therefore achieves the maximum price that can be expected so long as each bidder (or, equivalently, each potential buyer) has a non-negative expected profit.

Proof: The proof first characterizes the λ that maximizes the net value, and then shows that at this λ , bidders have an expected net profit of exactly zero. To start, note that the market value equals

$$\int_z \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \int_x x n F^{n-1}(x) dF(x) dH(z) \text{ and the equilibrium price equals}$$

$$\int_z \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \int_x x n(n-1) (1-F(x)) F^{n-2}(x) dF(x) dH(z). \text{ Then, to}$$

maximize the net value, must satisfy the first order condition that the

market value less the bidders' costs of $c\lambda$, all differentiated with

respect to λ , equals zero. This gives that $\lambda c, = \int_z \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{h!} (n-\lambda)$

$\int_x x F^{n-1}(x) dF(x) dF(z)$. Now, in total, the bidders' expected profit

less their expected costs equals the market value less the expected

price and the bidders' costs $c\lambda$, which in turn equals

$$\begin{aligned} & \int_z \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left[\int_x x n F^{n-1}(x) dF(x) - \int_x x n(n-1) (1-F(x)) F^{n-2} dF(x) \right. \\ & \quad \left. - n(n-\lambda) \int_x x F^{n-1}(x) dF(x) \right] dH(z) \\ &= \int_z \sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \left[-n(n-1) \int_x x F^{n-2}(x) dF(x) + n\lambda \int_x x F^{n-1}(x) dF(x) \right] dH(z) \\ &= \int_z \int_x x \left[\sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} n\lambda F^{n-1}(x) - \sum_{n=2}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} n(n-1) F^{n-2}(x) \right] dF(x) dH(z) \\ &= \int_z \int_x x \left[\sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^2 \lambda^{n-1}}{(n-1)!} F^{n-1}(x) - \sum_{n=2}^{\infty} \frac{e^{-\lambda} \lambda^2 \lambda^{n-2}}{n!} F^{n-2}(x) \right] dF(x) dH(z) \\ &= \int_z \int_x x \lambda^2 e^{-\lambda} \left[\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} F^m(x) - \sum_{m=2}^{\infty} \frac{\lambda^m}{m!} F^m(x) \right] dF(x) dH(z) \end{aligned}$$

which equals zero. Thus the bidder's have a net expected profit of zero at the λ which maximizes the net value, and the equilibrium price must therefore equal the maximum possible net value.

Possible Improvements

So far, the oral auction mechanism has been independent of the various details of the model. Now, we consider improvements possible to this basic mechanism if it may be tailored to the specific instance--that is, if the auction mechanism may depend on p , the c_j 's, and the distribution of the bidders' values. Specifically, we examine the improvements possible through employing an appropriate fee (paid by potential buyers or actual bidders to the seller), reservation price, or subsidy (paid by the seller to potential buyers or actual bidders).

Theorem 7: No mechanism that guarantees bidders a non-negative expected profit can generate a higher expected selling price than the equilibrium in our model if the seller charges each actual bidder i , an entry fee of $(V(n^*) - V(n^*-1))/p - c_i$, or, alternatively, charges each potential buyer fee of $V(n^*) - V(n^*-1) - c_i p$, where n^* denotes the equilibrium number of potential buyers.

Proof: Either type of charge reduces the expected profit of each potential buyer to exactly zero. This effects neither the equilibrium number of potential buyers nor the bidding of actual bidders at equilibrium. However, the expected selling price plus the fees paid to the seller now equal the net value at the equilibrium number of potential buyers, which equals the maximum possible net value.

For example, for independent, uniform $(0,1)$ distributed private values, with $p = 1$, the market value $V(n) = n/(n+1)$, the equilibrium price for $r = 0$ is $(n-1)/(n+1)$, and the bidders' combined expected profit net of participation costs equals $1/(n+1)$. Thus, for a participation cost c of .04 for each bidder, the equilibrium number of bidders will be four; for $n = 4$, $V(n) - V(n-1) = .05 > .04$, while for $n=5$, $V(n) - V(n-1) = .0333... < .04$. The equilibrium price would be $3/5$, and bidders should expect a combined profit (net of participation costs) equal to $1/5 - 4(.04) = .04$. Thus, charging each bidder an entry fee of .01 both reduces the bidders' expected profit to zero and increases the seller's expected revenue to $.6 - 4(.01) = .64$, which is the maximum that the net value $n/(n-1) - .04n$ can be.

Thus, with an appropriately set entry fee, the seller can extract the full maximum possible net value of the object. Note that the mechanism extracts the full value without exploiting any dependence among bidders' x_j 's, as, for instance, Myerson (1981) does in his example of a full value extracting auction. However, as with many "optimal" auctions, both the optimality of the auction and parametrisation of the optimal auction itself depends heavily on details (such as the actual distribution of the bidders' values) of the model.

As an alternative to entry fees, we now consider the possibility of a non-trivial reservation price. As noted in the introduction, an appropriately chosen reservation price increases the expected equilibrium selling price in the case of a fixed number (or distribution) of bidders. However, it also decreases each bidder's expected

profit, and might therefore decrease the number of bidders. This reduction in the number of bidders may more than offset any gain from a non-trivial reservation price--Engelbrecht-Wiggans (1986) provides an example illustrating this possibility.

In particular, let $N(r)$ denote the random number of actual bidders in an auction with reservation price r . Let $p_n(r)$ denote the probability that $N(r)$ equals n ; assume that $\sum_{n=0}^{n=\infty} p_n(r) = 1$ for all $r \geq 0$. Note that $N(r)$ may be concentrated at a single n , thereby giving the case of a deterministic number of bidders.

We say that a reservation price r_1 results in more bidders than a reservation price of r_2 ($r_2 > r_1 \geq 0$) if $\sum_{n=0}^{n=k} p_n(r_2) \geq \sum_{n=0}^{n=k} p_n(r_1)$ for all k , with a strict inequality for at least one k . In words, one reservation price generates more bidders than another if the distribution of the number of bidders under the smaller reservation price stochastically dominates the number of bidders under the larger reservation price.

Theorem 8: In the case of independent private values, if a reservation price of zero results in a greater number of bidders than a reservation price of r , a reservation price of zero generates at least as high an equilibrium selling price in the oral auction as a reservation price of r .

Proof: Define $q_m(r,n)$ --the conditional probability of a non-negative integer random variable $M(r,n)$ being equal to m given $N(r)$ equals n --so that $N(r) + M(r,n)$ has the same distribution as

$N(0)$; the assumed stochastic dominance assures that such probabilities can be defined. Now, start with an oral auction with a reservation price of r . If none of the n bidders bid at least r , conduct a second oral auction, now with a reservation price of zero, and with m new bidders (and none of the previous bidders). Clearly, each bidder bidding truthfully would still give an equilibrium, and therefore, this (possibly) two stage mechanism generates at least as high an expected equilibrium selling price as the original one stage oral auction with reservation price r .

Now, we want to view this (possibly) two stage auction as a single stage auction. To do so, imagine that Nature first selects a value n for $N(r)$ and then selects a value m for $M(r,n)$ (conditional on $N(r) = n$). Now set the actual number of bidders at $n + m$. Each bidder sees his private value for the object and then bids. Then, randomly label n of the bidders "type 1" and label the remaining m bidders "type 2." If at least one type 1 bidder bids r or more, the highest type 1 bidder wins the object and pays the larger of the second highest type 1 bid and the reservation price r . Otherwise, the highest type 2 bidder (if any) wins the object and pays the higher of the second highest type 2 bid (if there is one) or zero. In this auction, each bidder has the dominant strategy of bidding equal to his known private value for the object, and the expected equilibrium selling price will be equal to that in the (possibly) two stage mechanism.

Then, the revenue equivalence result of Myerson (1981) assures that for any fixed $n + m$ (known to the bidders), this latest one stage

an auction mechanism will generate the same expected selling price at equilibrium as a mechanism with the same allocation rule as before, but in which the price paid by the winner (if any) equals the second highest of all the bids or zero if there are no other bids; possibly, the winner pays an amount equal to, or even greater than, his own bid. At equilibrium in this latest mechanism, each bidder must bid strictly less than his own value with some positive probability, and never bid greater than his own value.

Finally, compare the expected equilibrium selling price in this latest mechanism--an expected price that at least equalled that of the oral auction with a reservation price of r and a random number $N(r)$ of bidders--to the oral auction with a reservation price of zero and a random $N(0)$ of bidders (the same number as in the previous mechanisms). All we change is the allocation rule; now the highest bidder (if there is any) wins. As a result, however, each bidder can once again bid truthfully. This results in bids at least as high as in the previous mechanism--and often strictly greater. Therefore, the second largest bid will be at least as large as before. Thus, at equilibrium, this final oral auction with a reservation price of zero (and a random number $N(0)$ of bidders) generates at least as great an expected selling price as the oral auction with a reservation price of r (and a random number $N(r)$ of bidders).

Note that the theorem did not depend on many of the details of our model. In particular, it does not matter how the distribution of the number of actual bidders arises. Nor need all bidders have the same

c_j (it doesn't enter into the proof) or the same probability p_j of bidding.

As a corollary to this theorem we may conclude that in our oral auction model, the optimal reservation price is the one that maximizes the selling price without reducing the number of potential buyers. Thus, typically, the seller should increase the reservation price from zero until either the bidder's expected net profit shrinks to zero, or until the reservation price equals the optimum for a fixed number of bidders equal to the equilibrium number for a reservation price of zero. In no case should the seller select a reservation price so large as to decrease the number of potential buyers.

The larger the optimum reservation price in our oral auction model, the greater the chance that the seller keeps the object. Also, the larger the reservation price can be without driving away a potential buyer, the larger an expected profit the bidders' must have had, and the smaller a subsidy would be required to attract another bidder if the reservation price were set at zero. In fact, in the extreme case, an arbitrarily small subsidy would attract another potential buyer, the potential buyers would have an expected net profit of zero, and the expected equilibrium selling price would equal the maximum net value possible if the potential buyers' costs had just been a smidgeon smaller. This suggests that an appropriately sized subsidy might do better for the seller than the optimal reservation price.

Theorem 9: In the case of a zero reservation price, the optimal subsidy (paid either to all potential buyers, to only those potential buyers who require a subsidy in order to have a non-negative expected profit, or to all actual bidders) will never be so large so as to increase the number of potential buyers more than one above the original equilibrium number.

Proof: If the number of potential buyers increases as a result of a subsidy, then the subsidy should be set such that the last additional potential buyer has a zero expected profit; otherwise a smaller subsidy would have had a similar effect on the number of potential buyers, but at a smaller cost to the seller. Imagine starting with a subsidy just large enough to attract one additional potential buyer. The potential buyers will have some expected net profit (presumably zero if all potential buyers have the same c_j). The seller receives an expected net revenue equal to the net value of the object less this expected profit. However, when the subsidy becomes large enough to attract yet one more potential buyer, the net value drops and the seller's expected revenue--the net value less the subsidies--must also drop. Thus, the subsidy should never be larger than needed to just barely increase the number of potential buyers to one over the equilibrium number.

We close with an example to illustrate optimal reservation prices and subsidies, and to illustrate that sometimes a reservation price should be preferred to a subsidy, and sometimes vice versa. In particular, consider the case of independent private values distributed

uniformly on the unit interval. Assume that all potential participants have the same c_j (say c), and assume that p equals unity. Then without a reservation price or subsidy, the n bidders would have an expected equilibrium profit of $1/(n+1)$ gross of participation costs. Thus, for $1/12 < c \leq 1/6$, the equilibrium number of bidders will be two.

The bidding equilibrium to this example with a reservation price of r generates an expected price of $r^n + ((n+1) - 2nr^{n+1})/(n+1)$, and an expected total profit to the n bidders (gross of participation costs) of $-r^n + (nr^{n+1} + 1)/(n+1)$. For a c just a hair above $1/12$, a very small subsidy (and zero reservation price) would result in an equilibrium with three bidders, an expected price of $2/4 = 1/2$, and an expected net revenue to the seller of just under $1/2$. On the other hand, for zero subsidy and a reservation price of just under $1/2$, two bidders would have a combined expected profit (gross of participation costs) of just over $1/6$ --just enough to cover the participation costs of two bidders. Thus, as c drops to $1/12$, the optimal reservation price rises to $1/2$. But even for reservation price of $1/2$, the expected price from two bidders would be only $5/12$ --strictly less than the just under $1/2$ that can be obtained from an appropriate subsidy. In short, as c drops to $1/12$, an optimal subsidy and no reservation price results in a greater expected equilibrium revenue to the seller than that possible from an optimal reservation price and no subsidy.

On the other hand, consider the case of $c = 5/36$. Now a reservation price of $1/4$ (and no subsidy) would give two bidders a combined expected profit $54/192 = 162/576$ --more than enough to cover

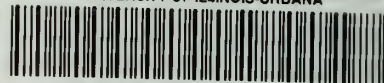
their participation costs of $2(5/36) = 160/576$. Thus, a reservation price of $1/4$ (and no subsidy) would result in an equilibrium with two bidders and an expected price of $3/8$. But, to get three bidders would require a subsidy of $3(5/36) - 1/4$ (the bidders' expected profit when $n=3$) $= 1/6$. Three bidders would give rise to an expected equilibrium price of $1/2$. Net of the subsidy, the seller could expect a revenue of $1/2 - 1/6 = 1/3$ --strictly less than the $3/8$ possible with a reservation price of $1/4$ and no subsidy. Here, when $c = 5/36$, even a sub-optimal reservation price and no subsidy does better for the seller than an optimal subsidy with no reservation price; in fact this will be the case for $.1160256 < c \leq 1/6$, while the reverse is true for $1/12 < c < .1160255$. Thus a small change in one parameter of the model may swing the seller from preferring a subsidy over a reservation price to the other way around.

Despite this inconclusiveness, we can conclude something of interest from these last three theorems. In particular, even if the seller uses an entry fee, a reservation price, or a subsidy to custom tailor the basic oral auction to a specific situation, the resulting number of potential buyers need never be less than the original equilibrium number, nor need it ever exceed the original equilibrium by more than one. Thus, roughly speaking, except for the discreteness of the number of potential buyers, our oral auction without reserve generates as high an expected equilibrium selling price as can be reasonably expected in cases of risk neutral bidders with private values, and non-zero costs of participation.

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